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On the global solvability of Kirchhoff equation for non-analytic initial data

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Abstract

We give sufficient conditions for the global solvability of Kirchhoff equation in terms of the spectral resolutions of the initial data $u(0,x)$, $u_t(0,x)$. We assume no smallness conditions and only “Sobolev-type” regularity.

MSC: 35L70

Keywords: Global solvability; Kirchhoff equation

1. Introduction

We study the global solvability of the problem

$$u_{tt} - m\left(\int_\Omega |\nabla u(x,t)|^2\, dx\right)\Delta u = 0 \quad \text{in } \Omega \times [0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

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where Ω is the whole space $\mathbb{R}^n$ $(n \geqslant 1)$, or a bounded C^2 domain and $u(x,t)$ satisfies the Dirichlet boundary condition $u|_{\partial\Omega} = 0$. Assuming Eq. (1.1) is strictly hyperbolic, namely

$$m(s) \geqslant \delta > 0 \quad \forall s \geqslant 0, \tag{1.3}$$

we give sufficient conditions for the existence of global *strong* solutions to problem (1.1) and (1.2) in terms of the spectral resolution of the initial data (u_0, u_1). In particular, we do not require any smallness conditions and we assume only “Sobolev-type” regularity. See Theorem 2 and Remark 3 below. To begin with, let us consider problem (1.1) and (1.2) in the case $\Omega = \mathbb{R}^n$:

Definition 1. We say that a solution $u(x,t)$ of problem (1.1) and (1.2) is a *strong solution* in $\mathbb{R}^n \times [0,T)$ if $u(x,t) \in C^k\left([0,T); H^{2-k}\,(\mathbb{R}^n)\right)$ for $0 \leqslant k \leqslant 2$. If $T = +\infty$ we say that $u(x,t)$ is a *global* strong solution.

Theorem 1. *Given $u_0, u_1 \in L^2(\mathbb{R}^n)$ and $m(s) \in C^2$ satisfying (1.3), let us suppose that there exist a sequence of positive numbers $\{\rho_j\}_{j \geqslant 0}$, $\rho_j \to \infty$, and $\eta > 0$ such that*

$$\sup_j \int_{|\xi| > \rho_j} \left[|\xi|^4 \, |\hat{u}_0(\xi)|^2 + |\xi|^2 \, |\hat{u}_1(\xi)|^2 \right] \frac{e^{\eta \rho_j^2 / |\xi|}}{\rho_j^2} \, d\xi < \infty. \tag{1.4}$$

Then the Cauchy problem (1.1) and (1.2) has a unique global strong solution. Furthermore, the same conclusion holds true assuming $m(s) \in C^3$ and $u_0, u_1 \in L^2(\mathbb{R}^n)$ such that

$$\sup_j \int_{|\xi| > \rho_j} \left[|\xi|^5 \, |\hat{u}_0(\xi)|^2 + |\xi|^3 \, |\hat{u}_1(\xi)|^2 \right] \frac{e^{\eta \rho_j^3 / |\xi|^2}}{\rho_j^3} \, d\xi < \infty. \tag{1.5}$$

Remark 1. If (1.4) holds then $(u_0, u_1) \in H^2 \times H^1$. Condition (1.5) implies that $(u_0, u_1) \in H^{5/2} \times H^{3/2}$. In this case the solution belongs to $C^k\left([0,\infty); H^{5/2-k}(\mathbb{R}^n)\right)$ for $0 \leqslant k \leqslant 2$ (see [1,2,15,21–23]). More precisely, from the proof of Lemma 2.2 below, it will be clear that if the initial data (u_0, u_1) satisfy (1.4) (resp., (1.5)) then $(u(\cdot,t), u_t(\cdot,t))$ satisfies (1.4) (resp., (1.5)) for all $t \geqslant 0$.

Let us recall that, in the case $n = 1$ and with $m(s)$ a positive C^1 function, the first result of global solvability of Kirchhoff equation was established by Bernstein in [4] for periodic analytic data. For $n \geqslant 1$ Pohožaev [18] (see also [3,5,10]) proved the global solvability of problem (1.1) and (1.2) assuming $(u_0, u_1) \in H_\Delta$

where

$$H_\Delta \stackrel{\text{def}}{=} \left\{ (u_0, u_1) \;\middle|\; \overline{\lim}\; k \left[\left\| \Delta^{k+\frac{1}{2}} u_0 \right\|_{L^2}^2 + \left\| \Delta^k u_1 \right\|_{L^2}^2 \right]^{-\frac{1}{4k}} > 0 \right\} \tag{1.6}$$

for $k = 2^j$. Later, Nishihara [12] (see also [9,13,24]) proved the global solvability for L^2 initial data satisfying a *quasi-analytic* condition: $(u_0, u_1) \in H_Q$ where

$$H_Q \stackrel{\text{def}}{=} \left\{ (u_0, u_1) \;\middle|\; \int \left[|\xi|^2 \, |\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2 \right] e^{\eta|\xi|/\ln(1+|\xi|)} \, d\xi < +\infty \right\} \tag{1.7}$$

for some $\eta > 0$. Note that $\mathcal{A}^2_{L^2} \subset H_\Delta, H_Q \subset H^\infty$, where $\mathcal{A}_{L^2}$ is the space of real analytic functions f such that $\|\partial^\alpha f\|_{L^2} \leqslant C\gamma^{|\alpha|}\,\alpha!$ for all $\alpha \in \mathbb{N}^n$, for some constants $C,\ \gamma \geqslant 0$ (see also (4.8)) and $H^\infty = \bigcap_k H^k$. Thus, in [4,18,12] the global solvability is proved only for *very regular* data. In [11], assuming $m(s) \in C^2$, the global solvability of (1.1) and (1.2) was proved for initial data $u_0, u_1 \in L^2(\mathbb{R}^n)$ such that, for a suitable $\left\{\rho_j\right\}_{j \geqslant 0}, \rho_j \to \infty$, and $\eta > 0$

$$\sup_j \int_{|\xi|>\rho_j} \left[|\xi|^4 \, |\hat{u}_0(\xi)|^2 + |\xi|^2 \, |\hat{u}_1(\xi)|^2 \right] e^{\eta\rho_j^2/|\xi|} \, d\xi < \infty. \tag{1.8}$$

Since the weight function $\exp\left\{\eta\rho_j^2/|\xi|\right\}$ of condition (1.8) is stronger than the weight $\rho_j^{-2} \exp\left\{\eta\rho_j^2/|\xi|\right\}$ in condition (1.4) (see Theorem 2, below), Theorem 1 improves the result of Manfrin [11]. Moreover, observe that there are initial data (u_0, u_1), satisfying (1.4) (or (1.8)), such that $(u_0, u_1) \notin H^{2+\varepsilon} \times H^{1+\varepsilon}$ for every $\varepsilon > 0$. See Remark 2 below. Hence, we have global solvability of problem (1.1) and (1.2) even for suitable non-smooth data. More precisely, let us introduce the following sets of initial data:

Definition 2. We say that $(u_0, u_1) \in B_\Delta$ if (1.8) holds for some sequence $\left\{\rho_j\right\}_{j\geqslant 0}$, $\rho_j \to +\infty$, and $\eta > 0$. Similarly, we say that $(u_0, u_1) \in B'_\Delta$ (resp., B''_Δ) if condition (1.4) (resp., condition (1.5)) holds.

Then for the sets $B_\Delta, B'_\Delta, B''_\Delta, H_\Delta, H_Q$ we have

Theorem 2.

(i) $B_\Delta + B_\Delta = H^2 \times H^1$ *and* $B''_\Delta + B''_\Delta = H^{5/2} \times H^{3/2}$;
(ii) $\mathcal{A}^2_{L^2} \subset H_\Delta \subset B_\Delta$ *with strict inclusions*;
(iii) $B_\Delta \subset B'_\Delta$ *and* $B'_\Delta \cap \left(H^{5/2} \times H^{3/2}\right) \subset B''_\Delta$ *with strict inclusions*;
(iv) $H_Q \not\subset B'_\Delta$ *and* $H_\Delta \not\subset H_Q$.

Remark 2. Observe that $H_\Delta, B_\Delta, B'_\Delta, B''_\Delta$ are *not* vector spaces. Moreover, using a Paley–Wiener argument [14, Theorem XII] it is possible to show (see [11]) that these sets do not contain compactly supported functions. To give a non-trivial example of initial data $(u_0, u_1) \in B_\Delta$ it is sufficient to consider a sequence $\{\rho_j\}_{j\geqslant 0}$ such that $\rho_0 = 1, \rho_{j+1} \geqslant 2\rho_j^2$. Given $(f, g) \in H^2 \times H^1$, we set

$$\hat{u}_0(\xi) \stackrel{\text{def}}{=} \chi(\xi)\hat{f}(\xi), \quad \hat{u}_1(\xi) \stackrel{\text{def}}{=} \chi(\xi)\hat{g}(\xi), \tag{1.9}$$

where $\chi(\xi)$ is the characteristic function of the set $A = \left\{\xi \,\middle|\, \rho_j^2 \leqslant |\xi| \leqslant \rho_{j+1} \text{ for some } j \geqslant 0\right\}$. Then (u_0, u_1) satisfies (1.8) with the sequence $\{\rho_j\}_{j\geqslant 0}$ for every $\eta > 0$.

1.1. Extension to the mixed problem

Replacing Fourier transform with Fourier series expansion and applying exactly the same arguments of the proof of Theorem 1, we can prove a similar result for the mixed problem in the cylinder $\Omega \times [0, \infty)$ where Ω is a bounded region in $\mathbb{R}^n$ with C^2 boundary. More precisely, we shall now consider the initial value problem (1.1) and (1.2) in $\Omega \times [0, \infty)$ with the boundary condition

$$u(x, t)\big|_{\partial\Omega\times[0,\infty)} = 0. \tag{1.10}$$

Definition 3. We say that $u(x, t)$ is a *strong solution* in $\Omega \times [0, T)$ if

$$u(x, t) \in C^0\left([0, T); H^2(\Omega)\right) \cap C^1\left([0, T); H_0^1(\Omega)\right) \cap C^2\left([0, T); L^2(\Omega)\right).$$

If $T = +\infty$ we say that $u(x, t)$ is a *global* strong solution.

Let $\{w_i\}_{i\geqslant 1}$ be a complete orthonormal system of eigenfunctions for the operator $-\Delta$ in $L^2(\Omega)$, with $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Namely,

$$\begin{cases} \Delta w_i + \lambda_i^2 w_i = 0 & \text{in } \Omega, \\ w_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where $(w_i, w_j)_{L^2} = \delta_{ij}$, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$ and $\lambda_i \to +\infty$, with each λ_i repeated the number of times equal to the multiplicity of the eigenvalue λ_i^2.

Then, setting

$$u_0(x) = \sum_{i=1}^{\infty} a_i w_i(x), \quad u_1(x) = \sum_{i=1}^{\infty} b_i w_i(x) \tag{1.12}$$

with $a_i = (u_0, w_i)_{L^2}$ and $b_i = (u_1, w_i)_{L^2}$, we have the following:

Theorem 3. *Given $u_0, u_1 \in L^2(\Omega)$ and $m(s) \in C^2$ satisfying (1.3), let us suppose that there exist a sequence of positive numbers $\{\rho_j\}_{j \geqslant 0}$, $\rho_j \to \infty$, and $\eta > 0$ such that*

$$\sup_j \sum_{\lambda_i > \rho_j} \left[\lambda_i^4 |a_i|^2 + \lambda_i^2 |b_i|^2 \right] \frac{e^{\eta \rho_j^2/\lambda_i}}{\rho_j^2} \, d\xi < \infty. \tag{1.13}$$

Then problem (1.1), (1.2) and (1.10) has a unique global strong solution. Furthermore, the same conclusion holds true assuming $m(s) \in C^3$ and $u_0, u_1 \in L^2(\Omega)$ such that

$$\sup_j \sum_{\lambda_i > \rho_j} \left[\lambda_i^5 |a_i|^2 + \lambda_i^3 |b_i|^2 \right] \frac{e^{\eta \rho_j^3/\lambda_i^2}}{\rho_j^3} \, d\xi < \infty. \tag{1.14}$$

Remark 3. In [11], assuming $m(s) \in C^2$, the global solvability of (1.1), (1.2) and (1.10) was proved for initial data $u_0, u_1 \in L^2(\Omega)$ such that, for a suitable sequence of positive numbers $\{\rho_j\}_{j \geqslant 0}$, $\rho_j \to \infty$, and $\eta > 0$ the Fourier coefficients a_i, b_i satisfy

$$\sup_j \sum_{\lambda_i > \rho_j} \left[\lambda_i^4 |a_i|^2 + \lambda_i^2 |b_i|^2 \right] e^{\eta \rho_j^2/\lambda_i} \, d\xi < \infty. \tag{1.15}$$

If (1.15) holds we say that $(u_0, u_1) \in B_\Delta(\Omega)$. Similarly, we say that $(u_0, u_1) \in B'_\Delta(\Omega)$ (resp., $B''_\Delta(\Omega)$) if condition (1.13) (resp., condition (1.14)) holds. Then, by the same arguments of the proof of (i) of Theorem 2, we can show that

$$B_\Delta(\Omega) + B_\Delta(\Omega) = \left(H^2(\Omega) \cap H_0^1(\Omega) \right) \times H_0^1(\Omega) \tag{1.16}$$

and that $B''_\Delta(\Omega) + B''_\Delta(\Omega) = V^{5/2}(\Omega) \times V^{3/2}(\Omega)$ where, for $\alpha \geqslant 0$, $V^\alpha(\Omega) = \left\{ f \in L^2(\Omega) : \sum_i \lambda_i^{2\alpha} |c_i|^2 < \infty \right\}$ with $c_i = (f, w_i)_{L^2}$ the Fourier coefficients of f. Besides, we can also prove that

$$B_\Delta(\Omega) \subset B'_\Delta(\Omega) \quad \text{and} \quad B'_\Delta(\Omega) \cap \left(V^{5/2} \times V^{3/2} \right) \subset B''_\Delta(\Omega) \tag{1.17}$$

with strict inclusions. To this end it is sufficient to follow the proof of (iii) of Theorem 2 using Weyl's asymptotic formula for the eigenvalues of the Laplace operator with Dirichlet boundary conditions [20, p. 271]. More precisely, on a bounded

Jordan measurable domain Ω the eigenvalues, λ_i^2, of problem (1.11) satisfy

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^n} = \frac{\tau_n}{(2\pi)^n} |\Omega|, \tag{1.18}$$

where $N(\lambda)$ is the number of eigenvalues $\lambda_i^2 < \lambda^2$ (each λ_i^2 repeated a number of times equal to its multiplicity), τ_n is the volume of the unit ball in $\mathbb{R}^n$ and $|\Omega|$ is the Jordan measure of Ω.

2. Preliminaries

2.1. A priori estimates for the linearized equation

Consider the infinite system of linear oscillating equations of the form

$$v_{tt} + a(t)\,|\xi|^2 v = 0 \quad \text{for } t \in [0, T), \quad \xi \in \mathbb{R}^n, \tag{2.1}$$

where $0 < T < \infty$, $v = v(\xi, t)$; $a(t)$ is a real-valued function satisfying the conditions

$$a(t) \in C^2\left([0, T)\right), \quad a(t) \geqslant \delta > 0 \quad \forall t \geqslant 0. \tag{2.2}$$

Given C^1 functions $a_1(t), a_2(t), a_3(t)$ we introduce the energies

$$E(\xi, t) \stackrel{\text{def}}{=} \frac{1}{2} a(t)a_1(t)\,|\xi|^4\,|v|^2 + \frac{1}{2} a_1(t)\,|\xi|^2\,|v_t|^2, \tag{2.3}$$

$$\mathcal{E}(\xi, t) \stackrel{\text{def}}{=} E(\xi, t) + a_2(t)\,|\xi|^2 \Re\,\{\bar{v} v_t\} + a_3(t)\,|v_t|^2, \tag{2.4}$$

where $\Re$ denotes the real part. Using (2.1) it easily follows (see the appendix) that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\xi, t) = &\left[\frac{1}{2}\frac{d}{dt}\left(a(t)a_1(t)\right) - a(t)a_2(t)\right] |\xi|^4\,|v|^2 \\ &+ \left[\frac{1}{2}a_1'(t) + a_2(t)\right] |\xi|^2\,|v_t|^2 \\ &+ \left[a_2'(t) - 2a(t)a_3(t)\right] |\xi|^2 \Re\,\{\bar{v} v_t\} + a_3'(t)|v_t|^2. \end{aligned} \tag{2.5}$$

Now, we can choose the coefficients $a_1(t), a_2(t), a_3(t)$.

Lemma 2.1. *Let us suppose that $a(t)$ satisfies* (2.2). *Then, taking*

$$a_1(t) = \frac{C}{\sqrt{a(t)}}, \quad a_2(t) = \frac{C}{4}\frac{a'(t)}{a(t)^{3/2}} \quad \textit{with } C \in \mathbb{R} \tag{2.6}$$

and $a_3(t) \equiv 0$, *we have* $\mathcal{E}'(\xi, t) = a_2'(t)\,|\xi|^2 \Re\{\bar{v} v_t\}$. *If we assume* (2.6) *and* $a(t) \in C^3$, *then* $a_2(t) \in C^2$. *In this case, setting*

$$a_3(t) = \frac{a_2'(t)}{2a(t)} = \frac{C}{8a(t)}\,\frac{d}{dt}\left(\frac{a'(t)}{a(t)^{3/2}}\right), \tag{2.7}$$

we obtain $\mathcal{E}'(\xi, t) = a_3'(t)\,|v_t|^2$.

Remark. In the following we fix $C = 1$.

Proof. The proof is straightforward. In the case $a_3 \equiv 0$ the statement follows from (2.5) because (2.6) implies that

$$\begin{cases} \dfrac{1}{2}\,\dfrac{d}{dt}\,(a(t)a_1(t)) - a(t)a_2(t) = 0, \\ \dfrac{1}{2}\,a_1'(t) + a_2(t) = 0. \end{cases} \tag{2.8}$$

On the other hand, if $a(t) \in C^3$, assuming (2.6) and taking $a_3 = a_2'/2\,a$ we immediately obtain $\mathcal{E}'(\xi, t) = a_3'(t)\,|v_t|^2$. $\square$

Lemma 2.2. *Let* $a_1(t) = a(t)^{-1/2}$ *and assume that* $0 < T < \infty$. *Then, for every* $0 < \varepsilon < T, C > 1$ *there exists* $\rho = \rho(\varepsilon, C) > 0$ *such that*

$$\frac{E(\xi, 0)}{C} \leqslant E(\xi, t) \leqslant C\,E(\xi, 0) \quad \textit{in } [0, T - \varepsilon], \tag{2.9}$$

for $|\xi| \geqslant \rho(\varepsilon, C)$.

Proof. We prove the second inequality in (2.9), the proof of the first one is similar. To begin with, let $K > 0$ such that

$$|a_2(t)|, \quad |a_2'(t)| \leqslant K \quad \text{in } [0, T - \varepsilon]. \tag{2.10}$$

Then, having

$$|\xi|^3\,|v|\,|v_t| \leqslant \frac{\sqrt{a(t)}}{2}\,|\xi|^4\,|v|^2 + \frac{1}{2\sqrt{a(t)}}\,|\xi|^2\,|v_t|^2 \equiv E(\xi, t), \tag{2.11}$$

from Lemma 2.1 (with $a_3(t) \equiv 0$) for $|\xi| > 0$ we find

$$\frac{d}{dt}\left[E(\xi, t) + a_2(t)\,|\xi|^2 \Re\{\bar{v} v_t\}\right] \leqslant K\,|\xi|^2\,|\bar{v} v_t| \leqslant K\,\frac{E(\xi, t)}{|\xi|}. \tag{2.12}$$

Now, taking $\rho_1 = \Lambda K$ with $\Lambda > 1$, we have

$$|\xi| \geqslant \rho_1 \Rightarrow \left| a_2(t)\, |\xi|^2 \Re\, \{\bar{v} v_t\} \right| \leqslant \frac{E(\xi, t)}{\Lambda} \quad \text{for } 0 \leqslant t \leqslant T - \varepsilon. \tag{2.13}$$

Then, for $|\xi| \geqslant \rho_1$, we obtain

$$\begin{aligned} E(\xi, t) &\leqslant E(\xi, 0) - \left[a_2(\tau)\, |\xi|^2 \Re\, \{\bar{v} v_t\} \right]_0^t + \frac{K}{|\xi|} \int_0^t E(\xi, \tau)\, d\tau \\ &\leqslant \frac{\Lambda + 1}{\Lambda} E(\xi, 0) + \frac{1}{\Lambda} E(\xi, t) + \frac{K}{|\xi|} \int_0^t E(\xi, \tau)\, d\tau. \end{aligned} \tag{2.14}$$

Hence, by Gronwall's Lemma, it follows that

$$E(\xi, t) \leqslant \frac{\Lambda + 1}{\Lambda - 1} E(\xi, 0) \exp\left(\frac{\Lambda K}{(\Lambda - 1)|\xi|} t \right). \tag{2.15}$$

Thus, it is sufficient to take $\Lambda > 1$ such that $\sqrt{C} > \frac{\Lambda+1}{\Lambda-1}$ and $\rho_\varepsilon \geqslant \rho_1$ such that

$$|\xi| \geqslant \rho_\varepsilon \Rightarrow \frac{\Lambda K (T - \varepsilon)}{(\Lambda - 1)|\xi|} \leqslant \ln \left(\frac{\Lambda + 1}{\Lambda - 1} \right). \qquad \Box \tag{2.16}$$

2.2. A priori estimates of the nonlinear terms

Let $u(x, t)$ be a strong solution of problem (1.1) and (1.2) in $\mathbb{R}^n \times [0, T)$. The Fourier transform in the space variables $v(\xi, t) = \hat{u}(\xi, t)$ satisfies the ordinary equation

$$v_{tt} + m\,(s(t))\, |\xi|^2 v = 0, \quad \text{where } s(t) \stackrel{\text{def}}{=} \int |\xi|^2\, |v|^2\, d\xi. \tag{2.17}$$

Multiplying the equation in (2.17) by $\bar{v}_t$ and integrating over $\mathbb{R}^n_\xi$, we find the well-known identity

$$\int |v_t(\xi, t)|^2\, d\xi + \Phi \left(\int |\xi|^2\, |v(\xi, t)|^2\, d\xi \right) = H_0 \quad \forall t \in [0, T), \tag{2.18}$$

where $\|u_1\|_{L^2}^2 + \Phi\left(\|\nabla u_0\|_{L^2}^2\right) \stackrel{\text{def}}{=} H_0$ and $\Phi(s) \stackrel{\text{def}}{=} \int_0^s m(y)\, dy$. Having, by (1.3), $\Phi(s) \geqslant \delta s$ for $s \geqslant 0$, it follows that $v(\xi, t)$ satisfies the a priori estimate

$$\int |v_t(\xi, t)|^2\, d\xi + \delta \int |\xi|^2\, |v(\xi, t)|^2\, d\xi \leqslant H_0 \quad \forall t \in [0, T) \tag{2.19}$$

and that $s(t)$ is bounded by H_0/δ. Since $u(x,t)$ is a strong solution, the term $s(t)$ is a C^2 function such that

$$|s'(t)| \leqslant 2 \int |\xi|^2 \, |v| \, |v_t| \, d\xi, \tag{2.20}$$

$$|s''(t)| \leqslant 2 \int |\xi|^2 \, |v_t|^2 \, d\xi + 2\, m\,(s(t)) \int |\xi|^4 \, |v|^2 \, d\xi. \tag{2.21}$$

In Section 3 of this paper, we will give in details only the proof of the second part of Theorem 1, because the proof of the first one is similar and can be easily obtained following almost the same estimates.

Thus we will assume in the following $m(s) \in C^3$ and $u(x,t) \in C^k\left([0,T); H^{5/2-k}(\mathbb{R}^n)\right)$ for $0 \leqslant k \leqslant 2$. Then, using the equation in (2.17) it easily follows that $s(t)$ is a C^3 function with

$$|s'''(t)| \leqslant 8m\,(s(t)) \int |\xi|^4 \, |v| \, |v_t| \, d\xi + 2\, \left|m'\,(s(t))\right| \, |s'(t)| \int |\xi|^4 \, |v|^2 \, d\xi. \tag{2.22}$$

Besides, having $0 \leqslant s(t) \leqslant \frac{H_0}{\delta}$, we can fix a constant $M > 0$ such that

$$m(s),\ |m'(s)|,\ |m''(s)|,\ |m'''(s)| \leqslant M \quad \text{for } 0 \leqslant s \leqslant 1 + \frac{H_0}{\delta}. \tag{2.23}$$

Setting

$$a(t) \overset{\text{def}}{=} m\,(s(t))\,, \tag{2.24}$$

we define the coefficients $a_1(t),\ a_2(t),\ a_3(t)$ as in (2.6) and (2.7) with $C = 1$. Taking account of (1.3) and (2.20)–(2.23), a straightforward computation gives the following estimates:

$$|a_2(t)| \leqslant C_2(\delta, M) \int |\xi|^2 \, |v| |v_t| \, d\xi, \tag{2.25}$$

$$|a_3(t)| \leqslant C_3(\delta, M) \left[\int |\xi|^2 \, |v_t|^2 \, d\xi + \int |\xi|^4 \, |v|^2 \, d\xi + \left(\int |\xi|^2 \, |v| |v_t| \, d\xi \right)^2 \right], \tag{2.26}$$

$$|a_3'(t)| \leqslant C_4(\delta, M)\left[\int |\xi|^4\, |v|\, |v_t|\, d\xi + \left(\int |\xi|^2\, |v|\, |v_t|\, d\xi\right)^3\right.$$
$$\left. + \left(\int |\xi|^2\, |v_t|^2\, d\xi + \int |\xi|^4\, |v|^2\, d\xi\right) \int |\xi|^2\, |v|\, |v_t|\, d\xi\right] \tag{2.27}$$

for suitable constants $C_2(\delta, M), C_3(\delta, M), C_4(\delta, M)$ depending only on δ and the upper bound M introduced in (2.23).

To prove Theorem 1, we define the energy functions

$$F(\xi, t) \overset{\text{def}}{=} |\xi| E(\xi, t) = \frac{\sqrt{m(s(t))}}{2}\, |\xi|^5\, |v|^2 + \frac{|\xi|^3 |v_t|^2}{2\sqrt{m(s(t))}} \tag{2.28}$$

and, for $\rho \geqslant 0$,

$$F_\rho(t) \overset{\text{def}}{=} \int_{|\xi| > \rho} F(\xi, t)\, d\xi. \tag{2.29}$$

Using (1.3), (2.19) and (2.23) we have

Lemma 2.3. *Let $u(x, t)$ be a strong solution of problem* (1.1) *and* (1.2) *in $\mathbb{R}^n \times [0, T)$. Then, for all $t \in [0, T)$ and for all $\rho \geqslant 0$ the Fourier transform $v(\xi, t) = \hat{u}(\xi, t)$ satisfies the* a priori *estimates*:

$$\int |\xi|^k\, |v|\, |v_t|\, d\xi \leqslant \frac{\rho^{k-1} H_0}{2\sqrt{\delta}} + \int_{|\xi| > \rho} \frac{F(\xi, t)}{|\xi|^{4-k}}\, d\xi \quad (k \geqslant 1), \tag{2.30}$$

$$\int |\xi|^k\, |v|^2\, d\xi \leqslant \frac{\rho^{k-2} H_0}{\delta} + \frac{2}{\sqrt{\delta}} \int_{|\xi| > \rho} \frac{F(\xi, t)}{|\xi|^{5-k}}\, d\xi \quad (k \geqslant 2), \tag{2.31}$$

$$\int |\xi|^k\, |v_t|^2\, d\xi \leqslant \rho^k H_0 + 2\sqrt{M} \int_{|\xi| > \rho} \frac{F(\xi, t)}{|\xi|^{3-k}}\, d\xi \quad (k \geqslant 0). \tag{2.32}$$

Proof. Let us prove (2.30). To begin with, for $0 \leqslant t < T$ we have the inequalities:
(a) for all $k \geqslant 1$ and $\rho \geqslant 0$,

$$|\xi|^k |v|\, |v_t| \leqslant \frac{\rho^{k-1}}{2\sqrt{\delta}} \left(|v_t|^2 + \delta\, |\xi|^2 |v|^2\right) \quad \text{for } |\xi| \leqslant \rho;$$

(b) from the definition (2.28) of $F(\xi, t)$,

$$|\xi|^k |v(\xi,t)|\,|v_t(\xi,t)| \leqslant \frac{F(\xi,t)}{|\xi|^{4-k}} \quad \text{for } |\xi| > 0.$$

Then, applying (a) and (2.19) for $|\xi| \leqslant \rho$, (b) for $|\xi| > \rho$, we can estimate the left-hand side of (2.30). For all $\rho \geqslant 0$ and $t \in [0, T)$ we easily have

$$\begin{aligned}\int |\xi|^k \,|v|\,|v_t|\,d\xi &= \int_{|\xi|\leqslant\rho} |\xi|^k\,|v|\,|v_t|\,d\xi + \int_{|\xi|>\rho} |\xi|^k\,|v|\,|v_t|\,d\xi \\ &\leqslant \frac{\rho^{k-1}H_0}{2\sqrt{\delta}} + \int_{|\xi|>\rho} \frac{F(\xi,t)}{|\xi|^{4-k}}\,d\xi,\end{aligned}$$

provided $k \geqslant 1$. By similar arguments we deduce (2.31) and (2.32). □

Remark 2.4. In the proof of Lemma 2.3 it is enough to assume $m(s)$ a continuous function satisfying (1.3). In (2.32) we may take $M = \max_{0\leqslant s\leqslant H_0/\delta} m(s)$.

Now, from (2.25), (2.26) and (2.30)–(2.32) of Lemma 2.3, for all $\rho > 0$ and $0 \leqslant t < T$, we have the inequalities:

$$\begin{aligned}\left| a_2(t)\,|\xi|^3\,\Re\{\bar{v}v_t\}\right| &\leqslant |a_2(t)|\,\frac{F(\xi,t)}{|\xi|} \\ &\leqslant C_2(\delta,M)\left[\frac{\rho H_0}{2\sqrt{\delta}} + \int_{|\xi|>\rho}\frac{F(\xi,t)}{|\xi|^2}\,d\xi\right]\frac{F(\xi,t)}{|\xi|} \\ &\leqslant C_2(\delta,M)\left[\frac{\rho H_0}{2\sqrt{\delta}} + \frac{F_\rho(t)}{\rho^2}\right]\frac{F(\xi,t)}{|\xi|},\end{aligned} \tag{2.33}$$

$$\begin{aligned}|a_3(t)|\,|\xi|\,|v_t|^2 &\leqslant 2\sqrt{M}\,|a_3(t)|\,\frac{F(\xi,t)}{|\xi|^2} \\ &\leqslant 2C_3(\delta,M)\sqrt{M}\Bigg[\rho^2 H_0\,\frac{1+\delta}{\delta} + 2\left(\sqrt{M}+\frac{1}{\sqrt{\delta}}\right)\int_{|\xi|>\rho}\frac{F(\xi,t)}{|\xi|}\,d\xi \\ &\quad + \left(\frac{\rho H_0}{2\sqrt{\delta}} + \int_{|\xi|>\rho}\frac{F(\xi,t)}{|\xi|^2}\,d\xi\right)^2\Bigg]\frac{F(\xi,t)}{|\xi|^2} \\ &\leqslant 2C_3(\delta,M)\sqrt{M}\Bigg[\rho^2 H_0\,\frac{1+\delta}{\delta} + 2\left(\sqrt{M}+\frac{1}{\sqrt{\delta}}\right)\frac{F_\rho(t)}{\rho} \\ &\quad + \left(\frac{\rho H_0}{2\sqrt{\delta}} + \frac{F_\rho(t)}{\rho^2}\right)^2\Bigg]\frac{F(\xi,t)}{|\xi|^2}.\end{aligned} \tag{2.34}$$

Besides, from (2.27) and (2.30)–(2.32), for all $\rho > 0$ and $0 \leqslant t < T$ we have

$$
\begin{aligned}
\left|a_3'(t)\right| \, |\xi| \, |v_t|^2 \leqslant\ & 2\sqrt{M} \left|a_3'(t)\right| \frac{F(\xi,t)}{|\xi|^2} \\
\leqslant\ & 2\sqrt{M}\, C_4(\delta, M) \left\{ \frac{\rho^3 H_0}{2\sqrt{\delta}} + \int_{|\xi|>\rho} F(\xi,t)\, d\xi \right. \\
& + \left(\frac{\rho H_0}{2\sqrt{\delta}} + \int_{|\xi|>\rho} \frac{F(\xi,t)}{|\xi|^2}\, d\xi \right)^3 \\
& + \left[\rho^2 H_0 \frac{1+\delta}{\delta} + 2\left(\sqrt{M} + \frac{1}{\sqrt{\delta}}\right) \int_{|\xi|>\rho} \frac{F(\xi,t)}{|\xi|}\, d\xi \right] \\
& \left. \times \left(\frac{\rho H_0}{2\sqrt{\delta}} + \int_{|\xi|>\rho} \frac{F(\xi,t)}{|\xi|^2}\, d\xi \right) \right\} \frac{F(\xi,t)}{|\xi|^2} \\
\leqslant\ & 2\sqrt{M} C_4(\delta, M) \left\{ \frac{\rho^3 H_0}{2\sqrt{\delta}} + F_\rho(t) + \left(\frac{\rho H_0}{2\sqrt{\delta}} + \frac{F_\rho(t)}{\rho^2} \right)^3 \right. \\
& + \left[\rho^2 H_0 \frac{1+\delta}{\delta} + 2\left(\sqrt{M} + \frac{1}{\sqrt{\delta}}\right) \frac{F_\rho(t)}{\rho} \right] \\
& \left. \times \left(\frac{\rho H_0}{2\sqrt{\delta}} + \frac{F_\rho(t)}{\rho^2} \right) \right\} \frac{F(\xi,t)}{|\xi|^2}.
\end{aligned}
\tag{2.35}
$$

Finally, to simplify the estimates of the next session we define

$$
G(\xi,t) \stackrel{\text{def}}{=} \left[a_2(t)\, |\xi|^3\, \Re\{\bar{v} v_t\} + a_3(t)\, |\xi|\, |v_t|^2 \right],
\tag{2.36}
$$

$$
F_\alpha^\beta(t) \stackrel{\text{def}}{=} \int_{\alpha<|\xi|\leqslant\beta} F(\xi,t)\, d\xi \quad \text{for } 0 < \alpha \leqslant \beta.
\tag{2.37}
$$

Besides, we introduce the quantity

$$
K \stackrel{\text{def}}{=} \sup_j \int_{|\xi|\geqslant\rho_j} \left[\frac{\sqrt{M}}{2} |\xi|^5 |\hat{u}_0(\xi)|^2 + \frac{|\xi|^3 |\hat{u}_1(\xi)|^2}{2\sqrt{\delta}} \right] \frac{e^{\eta\rho_j^3/|\xi|^2}}{\rho_j^3}\, d\xi,
\tag{2.38}
$$

where $\{\rho_j\}_{j\geqslant 0}$ and $\eta > 0$ are the same of condition (1.5), thus we have $K < \infty$.

3. Proof of Theorem 1

Here we give in details the proof of the second part of Theorem 1, i.e. we prove the global solvability in the case $m(s) \in C^3$ and the initial data satisfies (1.5). In the case $m(s) \in C^2$ and (u_0, u_1) satisfies (1.4) the proof is similar. Furthermore, since the Cauchy problem (1.1) and (1.2) is well posed in the space of strong solutions (see [1,2,6–8,16–19,21–23]), it will be sufficient to prove that the energy

$$F_0(t) \stackrel{\text{def}}{=} \int F(\xi, t)\, d\xi \tag{3.1}$$

cannot "blow-up" in finite time. In fact, defining

$$T \stackrel{\text{def}}{=} \sup \left\{ \tau > 0 \;\middle|\; \exists!\, u(x,t) \text{ strong solution in } \mathbb{R}^n \times [0, \tau) \right\} \tag{3.2}$$

and assuming by contradiction that $T < +\infty$, we can prove that $F_0(t)$ is uniformly bounded in $[0, T)$ if the initial data (u_0, u_1) satisfies condition (1.5). Hence, by standard arguments, it is easy to extend the strong solution $u(x,t)$ to a larger stripe $[0, T') \times \mathbb{R}^n$ with $T' > T$. Clearly, this contradicts the definition of T and proves that problem (1.1) and (1.2) is globally solvable.

Taking account of these considerations, let $u(x,t)$ be the unique strong solution in the maximal stripe $\mathbb{R}^n \times [0, T)$. Having $(u_0, u_1) \in H^{5/2} \times H^{3/2}$, it follows that $u(x,t) \in C^k\left([0,T); H^{5/2-k}(\mathbb{R}^n)\right)$, for $0 \leqslant k \leqslant 2$, and that $s(t) \in C^3([0,T))$. Thus, setting $a(t) = m(s(t))$ as in (2.24), we can apply the energy estimates of the previous section. More precisely, we consider $u(x,t)$ in the stripe $\mathbb{R}^n \times [T-\varepsilon, T)$ for $0 < \varepsilon \leqslant T$. From (2.5) and (2.36) and the definition (2.28) of $F(\xi, t)$ we have

$$F(\xi, t) = F(\xi, T-\varepsilon) - \left[G(\xi, \tau)\right]_{T-\varepsilon}^{t} + \int_{T-\varepsilon}^{t} a_3'(\tau)\, |\xi|\, |v_t|^2\, d\tau, \tag{3.3}$$

where, applying (2.33)–(2.35), we can estimate the terms $G(\xi, t)$ and $a_3'(t)\, |\xi|\, |v_t|^2$. For $|\xi| \geqslant \rho \geqslant 1$ we easily see that

$$|G(\xi, t)| \leqslant C_5(\delta, M) \left[p\,(H_0) + p\left(\frac{F_\rho(t)}{\rho^3}\right)\right] \frac{\rho}{|\xi|}\, F(\xi, t), \tag{3.4}$$

$$\left| a_3'(t) \right|\, |\xi|\, |v_t|^2 \leqslant C_6(\delta, M) \left[q\,(H_0) + q\left(\frac{F_\rho(t)}{\rho^3}\right)\right] \frac{\rho^3}{|\xi|^2}\, F(\xi, t) \tag{3.5}$$

for suitable constants $C_5(\delta, M), C_6(\delta, M) > 0$ and

$$p(y) = y + y^2, \quad q(y) = y + y^3. \tag{3.6}$$

Now, assuming $H_0 > 0$ (if $H_0 = 0$ then $u(x,t) \equiv 0$), we set

$$\lambda = 4C_5(\delta, M)\,p(H_0) + 1 \tag{3.7}$$

and, recalling the definitions (2.29) and (2.37), we write

$$F_\rho(t) = F_\rho^{\lambda\rho}(t) + F_{\lambda\rho}(t). \tag{3.8}$$

Then, from (3.4) and (3.7), we have

$$|G(\xi,t)| < \frac{F(\xi,t)}{2} \quad \text{for } |\xi| \geqslant \lambda\rho, \quad \rho \geqslant 1 \tag{3.9}$$

and $t \in [T-\varepsilon, T)$ such that the quantities $F_\rho^{\lambda\rho}(t)$, $F_{\lambda\rho}(t)$ satisfy

$$\text{(a)}\ \frac{F_\rho^{\lambda\rho}(t)}{\rho^3} < \frac{H_0}{2}, \quad \text{(b)}\ \frac{F_{\lambda\rho}(t)}{\rho^3} < \frac{H_0}{2}. \tag{3.10}$$

To continue, we choose $\varepsilon > 0$ and $\tilde{\rho} \geqslant 1$ such that

$$\frac{\rho M}{2\delta}\left(\frac{H_0}{\sqrt{\delta}} + 2H_0\right)\varepsilon + \ln\left(\frac{4K+1}{H_0}\right) \leqslant \frac{\eta\rho}{\lambda^2} \quad \forall \rho \geqslant \tilde{\rho}, \tag{3.11}$$

$$4C_6\, q(H_0)\, \varepsilon \leqslant \frac{\eta}{2}, \tag{3.12}$$

where $0 \leqslant K < \infty$ is defined in (2.38) and $\eta > 0$ is the constant in condition (1.5). Besides, noting that $F_\rho(T-\varepsilon) \to 0$ as $\rho \to +\infty$, we may also suppose that

$$\frac{F_\rho^{\lambda\rho}(T-\varepsilon)}{\rho^3} \leqslant \frac{H_0}{4}, \quad \frac{F_{\lambda\rho}(T-\varepsilon)}{\rho^3} \leqslant \frac{H_0}{4} \quad \forall \rho \geqslant \tilde{\rho}. \tag{3.13}$$

Thanks to (3.13), for every $\rho \geqslant \tilde{\rho}$ the conditions of (3.10) are verified in some maximal right neighborhood of $T-\varepsilon$, say

$$\left[T-\varepsilon, \tilde{T}\right), \quad \text{where } \tilde{T} = \tilde{T}(\rho) \tag{3.14}$$

with $\tilde{T}(\rho)$ maximal, $T - \varepsilon < \tilde{T}(\rho) \leqslant T$. In the sequel we will prove that, if (u_0, u_1) satisfies (1.5), then $\tilde{T}(\rho_j) = T$ provided ρ_j is sufficiently large. To begin with, let us estimate $F_\rho^{\lambda\rho}(t)$. Observing that $F'(\xi,t)$ satisfies he elementary inequality

$$|F'(\xi,t)| \leqslant \frac{M}{2\delta}\,|s'(t)|\,F(\xi,t) \tag{3.15}$$

and taking $\rho(\varepsilon) \geqslant \tilde{\rho}$ according to Lemma 2.2, i.e. such that

$$F(\xi, T-\varepsilon) \leqslant 2F(\xi,0) \quad \text{for } |\xi| \geqslant \rho(\varepsilon), \tag{3.16}$$

we have

$$F(\xi,t) \leqslant 2F(\xi,0)\,\exp\left\{\frac{M}{2\,\delta}\int_{T-\varepsilon}^{t} |s'(\tau)|\,d\tau\right\} \tag{3.17}$$

for all $|\xi| \geqslant \rho(\varepsilon)$ and $t \in [T-\varepsilon, T)$. Now, having $\frac{F_\rho(t)}{\rho^3} \leqslant H_0$ in $[T-\varepsilon, \tilde{T})$, from (2.20) and (2.30) we find

$$\int_{T-\varepsilon}^{t} |s'(\tau)|\,d\tau \leqslant \rho\left(\frac{H_0}{\sqrt{\delta}} + 2H_0\right)\varepsilon \quad \text{for } t \in [T-\varepsilon, \tilde{T}). \tag{3.18}$$

Thus, taking $\rho = \rho_j$ with $\rho_j \geqslant \rho(\varepsilon)$, from (3.11) and the definition of K, we have

$$\begin{aligned}
\frac{F_{\rho_j}^{\lambda\rho_j}(t)}{\rho_j^3} &\leqslant \int_{\rho_j \leqslant |\xi| \leqslant \lambda\rho_j} 2F(\xi,0)\,\rho_j^{-3}\exp\left\{\frac{\rho_j M}{2\delta}\left(\frac{H_0}{\sqrt{\delta}} + 2H_0\right)\varepsilon\right\} d\xi \\
&\leqslant \frac{H_0}{4K+1}\int_{\rho_j \leqslant |\xi| \leqslant \lambda\rho_j} 2F(\xi,0)\rho_j^{-3}\,\exp\left\{\frac{\eta\rho_j^3}{|\xi|^2}\right\} d\xi \\
&\leqslant H_0\,\frac{2K}{4K+1} < \frac{H_0}{2}
\end{aligned} \tag{3.19}$$

for all $t \in [T-\varepsilon, \tilde{T})$. The estimate (3.19) means that, taking $\rho = \rho_j$ with $\rho_j \geqslant \rho(\varepsilon)$, the first condition in (3.10) is always verified as long as the second holds. Thus, it remains to prove that for $\rho = \rho_j$ sufficiently large (3.10)(b) holds for all $t \in [T-\varepsilon, T)$. Now, from (3.3)–(3.9), for every fixed $\rho \geqslant \tilde{\rho}$ as long as the conditions (3.10) are verified for $t \geqslant T-\varepsilon$ we find the inequality

$$F(\xi,t) \leqslant 3F(\xi, T-\varepsilon) + 4C_6\frac{\rho^3 q(H_0)}{|\xi|^2}\int_{T-\varepsilon}^{t} F(\xi,\tau)\,d\tau \tag{3.20}$$

for all $|\xi| \geqslant \lambda\rho$. Hence, using (3.12) and applying Gronwall's Lemma to (3.20), for $\rho_j \geqslant \rho(\varepsilon)$ and $t \in [T-\varepsilon, \tilde{T}(\rho_j))$ we find the estimate

$$F(\xi, t) \leqslant 3F(\xi, T-\varepsilon) \exp\left\{ \frac{\eta\rho_j^3}{2|\xi|^2} \frac{t-T+\varepsilon}{\varepsilon} \right\} \quad \text{for } |\xi| \geqslant \lambda\rho_j \tag{3.21}$$

Thus, it will be sufficient to choose $\tilde{\rho}(\varepsilon) \geqslant \rho(\varepsilon)$ such that

$$\sup_{\rho_j \geqslant \tilde{\rho}(\varepsilon)} \int_{|\xi| > \lambda\rho_j} F(\xi, T-\varepsilon)\, \rho_j^{-3}\, e^{\eta\rho_j^3/2\,|\xi|^2}\, d\xi < \frac{H_0}{6}. \tag{3.22}$$

To this end, let us take $\Lambda > 0$ such that $24\, e^{-\Lambda\eta/2} K \leqslant H_0$. Then, setting

$$A_j = \left\{ |\xi| > \lambda\rho_j, \ \frac{\rho_j^3}{|\xi|^2} \geqslant \Lambda \right\} \quad \text{and} \quad B_j = \left\{ |\xi| > \lambda\rho_j, \ \frac{\rho_j^3}{|\xi|^2} < \Lambda \right\}, \tag{3.23}$$

thanks to (2.38) and (3.16) for $\rho_j \geqslant \rho(\varepsilon)$ we have

$$\begin{aligned}
&\int_{|\xi| > \lambda\rho_j} F(\xi, T-\varepsilon)\, \rho_j^{-3}\, e^{\eta\rho_j^3/2\,|\xi|^2}\, d\xi \\
&\quad \leqslant \int_{A_j} 2F(\xi, 0)\, \rho_j^{-3}\, e^{\eta\rho_j^3/2\,|\xi|^2}\, d\xi + \int_{B_j} 2F(\xi, 0)\, \rho_j^{-3}\, e^{\eta\rho_j^3/2\,|\xi|^2}\, d\xi \\
&\quad \leqslant e^{-\Lambda\eta/2} \int_{A_j} 2F(\xi, 0)\, \rho_j^{-3}\, e^{\eta\rho_j^3/|\xi|^2}\, d\xi + \int_{B_j} 2F(\xi, 0)\, \rho_j^{-3}\, e^{\Lambda\eta/2}\, d\xi \\
&\quad \leqslant \frac{H_0}{12} + \int_{|\xi| > \lambda\rho_j} 2F(\xi, 0)\, \rho_j^{-3}\, e^{\Lambda\eta/2}\, d\xi.
\end{aligned} \tag{3.24}$$

The last integral in (3.24) tends to 0 as $\rho_j \to +\infty$. Hence, it is clear that condition (3.22) is verified provided $\tilde{\rho}(\varepsilon)$ is sufficiently large.

This means that for $j_0 \geqslant 0$ such that $\rho_{j_0} \geqslant \tilde{\rho}(\varepsilon)$, we must have $\tilde{T}(\rho_{j_0}) = T$, i.e. both the assumptions of (3.10) are verified in $[T-\varepsilon, T)$ taking $\rho = \rho_{j_0}$. In particular, by (3.8) we have $F_{\rho_{j_0}}(t) \leqslant \rho_{j_0}^3 H_0$ in $[T-\varepsilon, T)$. Thus, using the a priori estimate (2.19), we finally obtain

$$F_0(t) \leqslant \left(\frac{1}{2\sqrt{\delta}} + \frac{\sqrt{M}}{2\delta} \right) \rho_{j_0}^3 H_0 + \rho_{j_0}^3 H_0 \tag{3.25}$$

for all $t \in [T-\varepsilon, T)$. This concludes the proof of Theorem 1. $\square$

4. Proof of Theorem 2

(i) Given $(u_0, u_1) \in H^2 \times H^1$, let $\{\rho_j\}_{j\geqslant 0}$ be a sequence satisfying the conditions

$$\rho_0 = 2; \quad \rho_{j+1} \geqslant \rho_j^2 \quad \text{for } j \geqslant 0. \tag{4.1}$$

Taking

$$\chi(\xi) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \rho_{2j} \leqslant |\xi| \leqslant \rho_{2j+1} \text{ for some } j \geqslant 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

we define the functions $v_i(x), w_i(x)$ for $i = 0, 1$ by setting

$$\hat{v}_i(\xi) = \chi(\xi)\,\hat{u}_i(\xi), \quad \hat{w}_i(\xi) = \big(1 - \chi(\xi)\big)\,\hat{u}_i(\xi). \tag{4.3}$$

Then, by the assumptions (4.1) and the definition of $\chi(\xi)$, it is easy to prove that (v_0, v_1) satisfies for every $\eta > 0$ the condition (1.8) with the subsequence $\{\rho_{2j+1}\}_{j\geqslant 0}$ instead of $\{\rho_j\}_{j\geqslant 0}$. In fact, for all $j \geqslant 0$, we have

$$\begin{aligned} &\int_{|\xi|\geqslant\rho_{2j+1}} \Big[|\xi|^4\,|\hat{v}_0(\xi)|^2 + |\xi|^2\,|\hat{v}_1(\xi)|^2\Big] \exp\Big\{\eta\rho_{2j+1}^2/|\xi|\Big\}\,d\xi \\ &\quad\leqslant \int_{|\xi|\geqslant\rho_{2j+2}} \Big[|\xi|^4\,|\hat{u}_0(\xi)|^2 + |\xi|^2\,|\hat{u}_1(\xi)|^2\Big] \exp\Big\{\eta\rho_{2j+1}^2/|\xi|\Big\}\,d\xi \\ &\quad\leqslant \int_{|\xi|\geqslant\rho_{2j+2}} \Big[|\xi|^4\,|\hat{u}_0(\xi)|^2 + |\xi|^2\,|\hat{u}_1(\xi)|^2\Big] \exp\{\eta\}\,d\xi \\ &\quad\leqslant \Big[\|\Delta u_0\|_{L^2}^2 + \|\nabla u_1\|_{L^2}^2\Big] \exp\{\eta\}, \end{aligned} \tag{4.4}$$

because $\chi(\xi) = 0$ for $\rho_{2j+1} < |\xi| < \rho_{2j+2}$. In the same way, but using the subsequence $\{\rho_{2j}\}_{j\geqslant 0}$ instead of $\{\rho_{2j+1}\}_{j\geqslant 0}$, we can see that (w_0, w_1) satisfies the condition (1.8). The proof that $B''_\Delta + B''_\Delta = H^{5/2} \times H^{3/2}$ is completely similar.

(ii) Condition (1.6) holds if and only if there exists a subsequence $\{k_j\}_{j\geqslant 0}$, $k_j = 2^{p_j}$ with $p_j \in \mathbb{N}$ and $p_j \to +\infty$, such that

$$\left\|\Delta^{k_j+\frac{1}{2}} u_0\right\|_{L^2}^2 + \left\|\Delta^{k_j} u_1\right\|_{L^2}^2 \leqslant \left(\frac{k_j}{\beta}\right)^{4k_j} \quad \text{for all } j \geqslant 0 \tag{4.5}$$

with $\beta > 0$ a suitable constant. Recalling the inequality

$$\frac{e^l}{e\sqrt{l}} \leqslant \frac{l^l}{l!} < e^l \quad \text{for all } l \geqslant 1 \tag{4.6}$$

and setting $l_j = 4k_j$, it easily follows that the condition (4.5) is equivalent to the following: there exist constants $C \geqslant 0$, $\gamma > 0$ such that

$$\int \left[|\xi|^2 \, |\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2\right] \frac{|\xi|^{l_j} \gamma^{l_j}}{l_j!} \, d\xi \leqslant C \quad \forall j \geqslant 0 \tag{4.7}$$

for a suitable sequence $\{l_j\}_{j \geqslant 0}$, with $l_j = 2^{p_j}$. This means that the first inclusion holds true, that is $\mathcal{A}^2_{L^2} \subset H_\Delta$, because

$$\mathcal{A}_{L^2} = \left\{ f \,\middle|\, \int |\hat{f}(\xi)|^2 \, e^{\mu |\xi|} \, d\xi < \infty \text{ for some } \mu > 0 \right\}. \tag{4.8}$$

To prove that $\mathcal{A}^2_{L^2} \neq H_\Delta$ we will see that there exist $f \in L^2$ and a sequence $\{l_j\}_{j \geqslant 0}$, $l_j = 2^{p_j}$ with $p_j \in \mathbb{N}$, such that

$$\int |\hat{f}(\xi)|^2 \, \frac{|\xi|^{l_j}}{l_j!} \, d\xi \leqslant 1 \quad \forall j \geqslant 0 \quad \text{but } f \notin \mathcal{A}_{L^2}. \tag{4.9}$$

To this end, we observe that by the second inequality of (4.6), for all $l \geqslant 1$

$$\frac{|\xi|^l}{l!} \leqslant 1 \quad \text{for} \quad |\xi| \leqslant l/e, \tag{4.10}$$

while, for $k \geqslant 2$,

$$\frac{|\xi|^l}{l!} \leqslant e^{|\xi|/k} \quad \text{for} \quad |\xi| \geqslant 2kl \, (1 + \ln k). \tag{4.11}$$

Then, we choose two increasing sequences: $\{k_j\}_{j \geqslant 0}$ such that $k_0 \geqslant 2$,

$$k_j < k_{j+1}, \quad k_j \to +\infty \tag{4.12}$$

and $\{l_j\}_{j \geqslant 0}$, of the form $l_j = 2^{p_j}$, such that $l_0 = 1$ and

$$\frac{l_{j+1}}{2e} \geqslant 2k_{j+1} l_j \left[1 + \ln k_{j+1}\right] \quad \forall j \geqslant 0 \tag{4.13}$$

and, finally, we take $f \in L^2$ such that

$$|\hat{f}(\xi)|^2 \stackrel{\text{def}}{=} \begin{cases} \dfrac{\lambda e^{-|\xi|/k_j}}{(1+|\xi|)^{n+1}} & \text{if } l_j/2e \leqslant |\xi| \leqslant l_j/e \text{ for some } j \geqslant 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.14}$$

with $\lambda > 0$ a suitable constant. From (4.14), for all $\mu > 0$ we have $\int |\hat{f}(\xi)|^2 e^{\mu|\xi|} d\xi = +\infty$, because $k_j, l_j \to +\infty$. Thus, $f \notin \mathcal{A}_{L^2}$. On the other hand, given a fixed integer $j \geqslant 0$ and writing

$$\int |\hat{f}(\xi)|^2 \frac{|\xi|^{l_j}}{l_j!} d\xi = \int_{|\xi| \leqslant l_j} |\hat{f}(\xi)|^2 \frac{|\xi|^{l_j}}{l_j!} d\xi + \int_{|\xi| > l_j} |\hat{f}(\xi)|^2 \frac{|\xi|^{l_j}}{l_j!} d\xi, \tag{4.15}$$

we can see that $f(x)$ satisfies the condition (4.9). In fact, the first integral is bounded by $\int_{|\xi| \leqslant l_j} \frac{\lambda d\xi}{(1+|\xi|)^{n+1}}$, because $\hat{f}(\xi) = 0$ for $l_j/e < |\xi| \leqslant l_j$ and (4.10) implies that $|\xi|^{l_j}/l_j! \leqslant 1$ for $|\xi| \leqslant l_j/e$. In the second integral, applying the inequality (4.11), for $l_i/2e \leqslant |\xi| \leqslant l_i/e$ with $i > j$ we have

$$\begin{aligned} |\hat{f}(\xi)|^2 \frac{|\xi|^{l_j}}{l_j!} &\leqslant \frac{\lambda e^{-|\xi|/k_i}}{(1+|\xi|)^{n+1}} \sup_{l_i/2e \leqslant |\xi| \leqslant l_i/e} \frac{|\xi|^{l_j}}{l_j!} \\ &\leqslant \frac{\lambda e^{-|\xi|/k_i}}{(1+|\xi|)^{n+1}} e^{|\xi|/k_i} = \frac{\lambda}{(1+|\xi|)^{n+1}}, \end{aligned} \tag{4.16}$$

because from (4.12) and (4.13) if follows that $l_i/2e \geqslant 2k_i l_j(1+\ln k_i)$. This means that (4.9) holds for all $j \geqslant 0$, provided $\lambda^{-1} \geqslant \int \frac{d\xi}{(1+|\xi|)^{n+1}}$. Finally, let us prove that if $(u_0, u_1) \in H_\Delta$, i.e. if (4.7) holds, then (u_0, u_1) satisfies (1.8) for $\eta < \gamma$ and $\{\rho_j\}_{j \geqslant 0}$ given by

$$\rho_j = \frac{l_j}{\gamma} = \frac{4k_j}{\gamma}. \tag{4.17}$$

In fact, there exist positive constants A, η such that

$$\frac{|\xi|^l \gamma^l}{l!} \geqslant A \exp\left\{\eta \frac{(l/\gamma)^2}{|\xi|}\right\} \quad \text{for all } |\xi| \geqslant l/\gamma, \tag{4.18}$$

because, taking $|\xi| = l/\gamma$ in inequality (4.18), we have only to verify that

$$\frac{l^l}{l!} \geqslant A e^{\eta l/\gamma} \tag{4.19}$$

and (4.19) follows from the first inequality of (4.6) for a suitable $A > 0$ provided $\eta < \gamma$. Thus we have proved that $H_\Delta \subset B_\Delta$ and (i) implies that the inclusion is strict because $H_\Delta \subset H^\infty$.

(iii) The inclusion $B_\Delta \subset B'_\Delta$ is clear. To prove that $B_\Delta \neq B'_\Delta$ it is sufficient to consider the sequence $\{\rho_j\}_{j\geqslant 0}$ given by $\rho_0 = 2$, $\rho_{j+1} = \frac{\rho_j^2}{\ln \rho_j}$ and $g \in L^2$ such that

$$|\xi|^4\, |\hat{g}(\xi)|^2 = \begin{cases} 0 & \text{if } |\xi| \leqslant 2 \text{ or } \rho_j \leqslant |\xi| \leqslant \dfrac{\rho_j^2}{2\ln \rho_j} \text{ for some } j \geqslant 0, \\ \dfrac{1}{|\xi|^n (\ln |\xi|)^2} & \text{otherwise.} \end{cases} \tag{4.20}$$

In fact, it is easy to see that $(g, 0) \in B'_\Delta$ because for $\eta \leqslant 1$ we have

$$\int_{|\xi| > \rho_j^2/2\ln\rho_j} |\xi|^4\, |\hat{g}(\xi)|^2\, \frac{e^{\eta\rho_j^2/|\xi|}}{\rho_j^2}\, d\xi \leqslant \frac{\omega_{n-1}}{\ln 2} \quad \forall j \geqslant 0,$$

where ω_{n-1} is the measure of S^{n-1}. On the other hand, for every $\rho \geqslant 2$, there exists $j \geqslant 0$ such that $\rho_j \leqslant \rho < \rho_{j+1}$. Then we have two possibilities

$$\rho_j \leqslant \rho < \frac{\rho_j^2}{2\ln\rho_j} \quad \text{or} \quad \frac{\rho_j^2}{2\ln\rho_j} \leqslant \rho < \frac{\rho_j^2}{\ln\rho_j} = \rho_{j+1}. \tag{4.21}$$

In the first case, provided ρ is large enough, for every $\eta > 0$ we have

$$\int_{|\xi| \geqslant \rho} |\xi|^4\, |\hat{g}(\xi)|^2\, e^{\eta\rho^2/|\xi|}\, d\xi \geqslant C\, \frac{\rho_j^\eta}{(\rho_j)^2} \tag{4.22}$$

for a suitable constant $C > 0$. In the second case, taking account that $\rho_{j+1}/2 \leqslant \rho < \rho_{j+1}$, we find

$$\int_{|\xi| \geqslant \rho} |\xi|^4\, |\hat{g}(\xi)|^2\, e^{\eta\,\rho^2/|\xi|}\, d\xi \geqslant C\, \frac{\rho_{j+1}^{\eta/4}}{(\ln \rho_{j+1})^2}. \tag{4.23}$$

Thus $(g, 0) \notin B_\Delta$. To prove that $B'_\Delta \cap \left(H^{5/2} \times H^{3/2}\right) \subset B''_\Delta$ it is enough to observe that

$$\frac{|\xi|^3}{\rho^3}\, \exp\left\{\eta\rho^3/2\,|\xi|^2\right\} \leqslant \frac{|\xi|^2}{\rho^2}\, \exp\left\{\eta\rho^2/\,|\xi|\right\}, \tag{4.24}$$

provided $\rho \leqslant |\xi| \leqslant \rho^{3/2}$ and $\rho > 0$ is large enough, i.e. such that $\rho^{1/2} \leqslant e^{\eta\rho^{1/2}/2}$. Finally, we can prove that the inclusion is strict following almost the same argument that we used to show that $B_\Delta \neq B'_\Delta$. More precisely, it is sufficient to choose $\{\rho_j\}_{j\geqslant 0}$ such that $\rho_0 = 2$, $\rho_{j+1} = 2\rho_j^{3/2}$ and $h \in L^2$ such that

$$|\xi|^5 \, |\hat{h}(\xi)|^2 = \begin{cases} 0 & \text{if } |\xi| \leqslant 2 \text{ or } \rho_j \leqslant |\xi| \leqslant \rho_j^{3/2} \text{ for some } j \geqslant 0, \\ \dfrac{1}{|\xi|^n \, (\ln|\xi|)^2} & \text{otherwise.} \end{cases} \tag{4.25}$$

Then we can easily verify that $(h, 0) \in B''_\Delta \setminus B'_\Delta$.

(iv) Let us prove that $H_Q \not\subset B'_\Delta$. In fact, given $w \in L^2$ such that $|\hat{w}(\xi)|^2 \geqslant (1 + |\xi|)^{-(n+1)}$, the data $(0, u_1)$ defined by

$$\hat{u}_1(\xi) \overset{\text{def}}{=} \hat{w}(\xi)\, e^{-|\xi|/2\ln(1+|\xi|)} \tag{4.26}$$

belongs to H_Q. On the other hand, for every fixed $\eta > 0$

$$\exp\left\{\frac{-|\xi|}{\ln(1+|\xi|)}\right\} \exp\left\{\frac{\eta\rho^2}{|\xi|}\right\} \geqslant \exp\left\{\frac{\eta\,|\xi|}{4}\right\}, \tag{4.27}$$

provided $\rho \leqslant |\xi| \leqslant 2\rho$ with $\rho > 0$ sufficiently large. Clearly, this means that $(0, u_1) \notin B'_\Delta$ and in the same way we see that $(0, u_1) \notin B'_\Delta$. Finally, let us show that $H_\Delta \not\subset H_Q$. From the second inequality in (4.6), for $\eta > 0$ sufficiently small, we have

$$\frac{|\xi|^l}{l!} \leqslant e^{\eta\,|\xi|^{1/2}} \quad \text{for all} \quad l \in \mathbb{N} \text{ and } |\xi| \geqslant \frac{l^4}{\eta^4}. \tag{4.28}$$

Taking account of this and setting

$$l_0 = 1, \quad l_{j+1} = 2^{4j+1} e\, l_j^4 \quad \text{for } j \geqslant 1, \tag{4.29}$$

we choose $f \in L^2$ such that

$$|\xi|^4 \, |\hat{f}(\xi)|^2 = \begin{cases} \dfrac{e^{-2^{-j}|\xi|^{1/2}}}{(1+|\xi|)^{n+1}} & \text{if } 2^{4j} l_j^4 \leqslant |\xi| \leqslant 2^{4j+1} l_j^4 \text{ for some } j \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.30}$$

Then, using (4.10) and (4.28) with $\eta = 2^{-j}$ and the condition (4.29), it follows that

$$\int |\xi|^4 \, |\hat{f}(\xi)|^2 \, \frac{|\xi|^{l_j}}{l_j!} \, d\xi \leqslant \int \frac{d\xi}{(1+|\xi|)^{n+1}}, \tag{4.31}$$

provided $j \geqslant 0$ is large enough. In fact, $\hat{f}(\xi) = 0$ for $l_j/e \leqslant |\xi| \leqslant 2^{4j} \, l_j^4$ and $|\xi|^{l_j}/l_j! \leqslant e^{2^{-k}|\xi|^{1/2}}$ for $2^{4k} l_k^4 \leqslant |\xi| \leqslant 2^{4k+1} l_k^4$, for $k \geqslant j$. On the other hand, for every $\eta > 0$ we have

$$|\xi|^2 \, |\hat{f}(\xi)|^2 \, \exp\left\{ \frac{\eta \, |\xi|}{\ln(1+|\xi|)} \right\} \geqslant \frac{\exp\{(\eta - 2^{-j}) \, |\xi|^{1/2}\}}{(1+|\xi|)^{n+3}} \tag{4.32}$$

provided $2^{4j} l_j^4 \leqslant |\xi| \leqslant 2^{4j+1} l_j^4$ for some $j \geqslant 0$, because $\ln\big(1+|\xi|\big) \leqslant |\xi|^{1/2}$ for $|\xi| \geqslant 1$. Consequently, for all $\eta > 0$ the integral

$$\int |\xi|^2 \, |\hat{f}(\xi)|^2 \, \exp\left\{ \frac{\eta \, |\xi|}{\ln(1+|\xi|)} \right\} d\xi \tag{4.33}$$

cannot converge. This proves that $(f, 0) \in H_\Delta$, but $(f, 0) \notin H_Q$.

Appendix A

Multiplying (2.1) by the factor $a_1(t) \, |\xi|^2 \, \bar{v}_t$, we easily obtain that

$$\frac{d}{dt}\Big(a_1(t) \, |\xi|^2 \, |v_t|^2 + a(t) a_1(t) \, |\xi|^4 \, |v|^2 \Big) = a_1'(t) \, |\xi|^2 \, |v_t|^2 + [a(t) a_1(t)]' \; |\xi|^4 \, |v|^2. \tag{A.1}$$

While, multiplying by the term $a_2(t) \, |\xi|^2 \, \bar{v}$ we find

$$\begin{aligned} \frac{d}{dt}\Big(a_2(t) \, |\xi|^2 \, \Re\{\bar{v} v_t\} \Big) \; &= \; -a(t) a_2(t) \, |\xi|^4 \, |v|^2 + a_2(t) \, |\xi|^2 \, |v_t|^2 \\ &\quad + a_2'(t) \, |\xi|^2 \, \Re\{\bar{v} \, v_t\}. \end{aligned} \tag{A.2}$$

Finally, using again Eq. (2.1), we have

$$\frac{d}{dt}\Big(a_3(t) |v_t|^2 \Big) = a_3'(t) \, |v_t|^2 - 2a(t) \, a_3(t) \, |\xi|^2 \, \Re\{\bar{v} \, v_t\}. \tag{A.3}$$

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